NAME:

Solutions to Math 150 Practice Exam 1.2

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. Calculate $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ [10 pts]

Solution: Method 1.

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{(\sqrt{x}+3)} = \frac{1}{6}$$

Method 2.

Simply observe that $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ is the definition of the derivative $\frac{d}{du}(\sqrt{u})|_{u=9}$.

Thus $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

2. Evaluate $\lim_{x\to 3} \frac{(x-1)(x-2)}{(x-3)}$ or explain why this limit doesn't exist.

[10 pts]

Solution: (i) $\lim_{x\to 3^-} \frac{(x-1)(x-2)}{(x-3)} = \frac{(2^-)(1^-)}{0^-} = -\infty$

(ii) $\lim_{x\to 3^+} \frac{(x-1)(x-2)}{(x-3)} = \frac{(2^+)(1^+)}{0^+} = +\infty$

Therefore the limit doesn't exist.

3. Evaluate $\lim_{x\to -\infty} 4x\left(3x-\sqrt{9x^2+1}\right)$ [10 pts]

Solution:

$\lim_{x\to -\infty} 4x\left(3x-\sqrt{9x^2+1}\right) =$

$$\lim_{x\to -\infty} 4x \frac{\left(3x-\sqrt{9x^2+1}\right)\left(3x+\sqrt{9x^2+1}\right)}{\left(3x+\sqrt{9x^2+1}\right)} = \lim_{x\to -\infty} 4x \frac{9x^2-(9x^2+1)}{3x+\sqrt{9x^2+1}}$$

$$= \lim_{x\to -\infty} 4x \frac{-1}{3x+\sqrt{9x^2+1}} = \lim_{x\to -\infty} \frac{-4x}{3x - x\sqrt{9+\frac{1}{x^2}}} = \frac{-4}{3-3^+} = -\infty$$

4. Compute $\lim_{x\to 0}\frac{\cos x-1}{sin^2 x}$ or explain why the limit doesn't exist.

[10 pts]

Solution: $\lim_{x\to 0}\frac{\cos x-1}{sin^2 x}=\lim_{x\to 0}-\frac{\cos x-1}{(\cos x-1)(\cos x+1)}=-1/2$

5. a) Find the derivative of $f(x)=5x^2-6x+1$ using the definition of the derivative at the point $a=2$. [5 pts]

Solution: Method 1.

$$f'(2)=\lim_{x\to 2}\frac{5x^2-6x+1-(5\cdot 2^2-6\cdot 2+1)}{x-2}=5\lim_{x\to 2}\frac{x^2-2^2}{x-2}$$
$$-6\lim_{x\to 2}\frac{x-2}{x-2}=5\lim_{x\to 2}(x+2)-6=20-6=14$$

Method 2.

$$f'(2)=\lim_{h\to 0}\frac{5(2+h)^2-6(2+h)+1-5\cdot 2^2+6\cdot 2-1}{h}=\lim_{h\to 0}\frac{20h+5h^2-6h}{h}=$$
14

b) Use the result in part (a) to write an equation of the line at the point $(a, f(a))$ [5 pt]

Solution: The tangent line has slope m = 14 and it passes through the point (2, 9). Thus we get the point slope form $y-9=14(x-2)$

6. Evaluate $\lim_{x\to 0}\frac{\tan 5x}{x}$ [10 pts]

Solution: $\lim_{x\to 0}\frac{\tan 5x}{x}=\lim_{x\to 0}\frac{\sin 5x}{x\cos 5x}=\lim_{x\to 0}\frac{1}{cos\ 5x}\frac{5\sin 5x}{5x}=5$

7. Show that the equation $x^3-5x^2+2x=-1$ has a solution. [10 pts]

Solution: Define $p(x)=x^3-5x^2+2x+1$. Then $p(x)$, being a polynomial is continuous everywhere on the real number line.
Observe: $p(0)=1>0$ and $p(1)=4-5=-1<0$. Thus by the Intermediate-Value Theorem, the equation $p(x)=0$ must have at least one root in the interval (0, 1)

8. Let $a > 0$ be a positive real number. Define $f(x) = \begin{cases} \sqrt{2x} & if \quad x < a \\ x & if \quad x \geq a \end{cases}$.

What is the value of a if f is continuous on the entire real number line? [10 pts]

Solution: The function $f(x)$ is made out of 2 continuous pieces. Hence a is the only point where the function could be discontinuous.
Since $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \sqrt{2x} = \sqrt{2a}$ and
$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} x = a$, we must have $\sqrt{2a} = a$ or $2a = a^2$. Therefore $a = 2$.

9. Compute $\lim_{x\to 0} \frac{\sqrt{1+5x}-\sqrt{1-5x}}{x}$ or explain why the limit doesn't exist.
[10 pts]

Solution: This limit is easily solved as soon as one recognizes that it is a derivative in disguise for the function $f(u) = \sqrt{u}$ at the point $u = 1$.

$$\lim_{x\to 0} \frac{\sqrt{1+5x}-\sqrt{1-5x}}{x} = \lim_{x\to 0} \frac{\sqrt{1+5x}-1+1-\sqrt{1-5x}}{x} =$$

$$\lim_{x\to 0}\left(\frac{\sqrt{1+5x}-1}{x} - \frac{\sqrt{1-5x}-1}{x}\right) =$$

$$\lim_{x\to 0} 5\left(\frac{\sqrt{1+5x}-1}{5x}\right) + \lim_{x\to 0} 5\left(\frac{\sqrt{1-5x}-1}{-5x}\right) =$$

$$5\frac{d}{du}(\sqrt{u})|_{u=1} + 5\frac{d}{du}(\sqrt{u})|_{u=1} = 5$$

10. Compute the derivative of $f(x) = \frac{(x^2-1)\sin x}{\sin x+1}$ [10 pts]

Solution: $f'(x) = \frac{(\sin x+1)[(2x)\sin x+(x^2-1)\cos x]-\cos x\,[(x^2-1)\sin x]}{(\sin x+1)^2}$

Extra-Credit

11. Prove by means of a delta-epsilon argument that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a}(f(x) + g(x)) = L + M$

[10 pts]

Solution: By assumption, there exist "counterfeit functions" $\delta_f(\epsilon)$ and $\delta_g(\epsilon)$ such that whenever $0 < |x - a| < \delta_f(\epsilon)$, $|f(x) - L| < \epsilon$ and whenever $0 < |x - a| < \delta_g(\epsilon)$, $|g(x) - L| < \epsilon$.

Let $\delta_{f+g}(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}$. Then, if $0 < |x - a| < \delta_{f+g}(\epsilon)$, $|x - a|$ is simultaneously smaller than $\delta_f\left(\frac{\epsilon}{2}\right)$ and $\delta_g\left(\frac{\epsilon}{2}\right)$. Hence we have

$$|f(x) + g(x) - L - M| = |(f(x) - L) + (g(x) - M)| \leq$$

$$|f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

12. Prove from the definition of the derivative that $\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{\frac{1}{n}-1}$

[10 pts]

Solution: $\frac{d}{dx}\left(x^{\frac{1}{n}}\right) = \lim_{z\to x} \frac{z^{\frac{1}{n}} - x^{\frac{1}{n}}}{z - x} =$

$$\lim_{z\to x} \frac{z^{\frac{1}{n}} - x^{\frac{1}{n}}}{\left(z^{\frac{1}{n}} - x^{\frac{1}{n}}\right)\left(\left(z^{\frac{1}{n}}\right)^{n-1} + \left(z^{\frac{1}{n}}\right)^{n-1}\left(x^{\frac{1}{n}}\right) + \cdots + \left(x^{\frac{1}{n}}\right)^{n-1}\right)} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} =$$

$$\frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$